ON THE INSCRIPTION OF REGULAR POLYGONS.

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Heretofore* it has been my aim to treat this geometric subject without the use of the customary complex imaginary. It is now my purpose to avoid the use of trigonometry also, basing my treatment solely upon algebraic and geometric principles.

The following two theorems, which are stated and proven geometrically in the last edition of Catalan's *Géométrie*, form the basis of the treatment:

Suppose a circle of unit radius divided at $A, A_1, A_2, A_3, \ldots, A_p, \ldots$ into $2p+1$ equal parts and the diameter AO drawn. Join O to $A_1, A_2, \ldots, A_p$. Let OA_r and OA_s (or briefly A_r and A_s) denote any two of these chords.

Theorem I. If $r + s \overline{\overline{<}} p, \quad A_r . A_s = A_{s-r} + A_{s+r}$. (1)

Theorem II. If $r + s > p, \quad A_r . A_s = A_{s-r} - A_{n-(s+r)}$. (2)

These follow at sight in their trigonometric form. Thus, (1) gives

$$2 \cos \frac{r\pi}{n} . \cos \frac{s\pi}{n} = \cos \frac{(s-r)\pi}{n} + \cos \frac{(s+r)\pi}{n},$$

where $n = 2p + 1$.

Corollary. $A_s^2 = 2 + A_{2s}$ if $2s \overline{\overline{<}} p$; but $= 2 - A_{n-2s}$ if $2s > p$. (3)

The following theorem is fundamental:

$$A_1 - A_2 + A_3 - A_4 + A_5 - \ldots - (-1)^p A_p = 1. \tag{4}$$

Proof. Suppose $A_1 - A_2 + A_3 - \ldots \pm A_p = x$.

Then
$$\begin{aligned} xA_1 &= A_1 (A_1 - A_2 + A_3 - \ldots \pm A_{p-2} \mp A_{p-1} \pm A_p) \\ &= 2 + A_2 - A_1 - A_3 + A_2 + A_4 - A_3 - A_5 + \ldots \\ &\quad \pm A_{p-3} \pm A_{p-1} \mp A_{p-2} \mp A_p \pm A_{p-1} \mp A_p, \end{aligned}$$

by applying (1) at every step of the multiplication except the first and last, when we apply (3) and (2) respectively.

$$\begin{aligned} \therefore\ xA_1 &= 2 - A_1 + 2(A_2 - A_3 + A_4 - A_5 + \ldots \mp A_{p-2} \pm A_{p-1} \mp A_p) \\ &= 2 + A_1 - 2(A_1 - A_2 + A_3 - A_4 + \ldots \pm A_{p-2} \mp A_{p-1} \pm A_p) \\ &= 2 + A_1 - 2x. \end{aligned}$$

$$\therefore\ (x-1)(2 + A_1) = 0.$$

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* An elementary sketch of my method appears in a series of articles (beginning Sept., 1894) in the American Mathematical Monthly.

But A_1 is not equal to -2. $\therefore x = 1$.

Compare the following direct trigonometric proof:

In the identity,

$$\cos a + \cos 3a + \cos 5a + \ldots + \cos (2p - 1) a = \frac{\sin 2pa}{\sin a},$$

let

$$a = \frac{\pi}{2p+1}.$$

Then

$$a = \angle AOA_1, \ 3a = \angle AOA_3, \text{ etc.}$$

Now

$$\cos (2p-1) a = \cos \frac{2p-1}{2p+1} \pi = - \cos \frac{2\pi}{2p+1};$$

$$\cos (2p-3) a = - \cos \frac{4\pi}{2p+1}; \text{ etc.}$$

Also

$$\sin 2pa = \sin \frac{2p\pi}{2p+1} = \sin \frac{\pi}{2p+1}.$$

$$\therefore \quad 2 \cos \frac{\pi}{2p+1} - 2 \cos \frac{2\pi}{2p+1} + 2 \cos \frac{3\pi}{2p+1} - 2 \cos \frac{4\pi}{2p+1} + \ldots$$

$$\pm 2 \cos \frac{p\pi}{2p+1} = 1.$$

To construct the equation whose roots are $A_1, -A_2, A_3, -A_4, \ldots, -(-1)^p A_p$.

$\Sigma A_i = 1$, where the plus sign is concealed in the root if i be odd and the minus sign if i be even. We may form the symmetric functions of the p roots, as follows:

$$(\Sigma A_i)^2 = \Sigma A_i^2 + 2 \Sigma A_i A_j.$$

But

$$A_i^2 = 2 + A_{2i}.$$

Hence

$$\Sigma A_i^2 = 2p + \Sigma A_{2i} = 2p - \Sigma A_i = 2p - 1.$$

$$\therefore \quad \Sigma A_i A_j = -(p-1).$$

$$(\Sigma A_i)^3 = \Sigma A_i^3 + 3 \Sigma A_i^2 A_j + 6 \Sigma A_i A_j A_k.$$

But

$$A_i^3 = A_i (2 + A_{2i}) = 3A_i + A_{3i}.$$

$$\therefore \Sigma A_i^3 = 3 \Sigma A_i + \Sigma A_{3i} = 4 \Sigma A_i = 4.$$

$$\Sigma A_i^2 A_j = \Sigma A_i^2 \Sigma A_j - \Sigma A_i^3 = (2p-1) - 4.$$

$$\therefore \Sigma A_i A_j A_k = -(p-2).$$

Whether we proceed by this method or the preferable ones below, it is necessary to find ΣA_i^m, denoted by s_m. We have found that

$$s_1 = 1\,, \quad s_2 = 2p - 1 = n - 2\,, \quad s_3 = 4\,.$$

$$A_i^4 = A_i(3A_i + A_{3i}) = 6 + 4A_{2i} + A_{4i}\,; \quad \therefore\ s_4 = 6p - 5 = 3n - 8\,.$$

$$A_i^5 = A_i(6 + 4A_{2i} + A_{4i}) = 10A_i + 5A_{3i} + A_{5i}\,; \quad \therefore\ s_5 = 16\,.$$

$$A_i^6 = 20 + 15A_{2i} + 6A_{4i} + A_{6i}\,; \quad \therefore\ s_6 = 20p - 22 = 10n - 32\,.$$

Similarly, $s_7 = 64$, $s_8 = 35n - 128$, $s_9 = 256$, $s_{10} = 126n - 512$.

THEOREM. *For the sum of like odd powers,* $s_{2k+1} = 2^{2k}$. (5)

$A_i^{2k-2} = (2 + A_{2i})^{k-1}$ gives on expansion a numerical term and terms linear in $A_{2i}, A_{4i}, A_{6i}, \ldots, A_{(2k-2)i}$, but no terms whose subscript is an *odd* multiple of i. Hence A_i^{2k-1} will contain no numerical term and no term whose subscript is an *even* multiple of i. We may therefore write

$$A_i^{2k-1} = aA_i + bA_{3i} + cA_{5i} + dA_{7i} + \ldots + gA_{(2k-1)i}\,.$$

Then $A_i^{2k+1} = (2 + A_{2i})\,A_i^{2k-1}$

$$= 2aA_i + 2bA_{3i} + 2cA_{5i} + \ldots + 2gA_{(2k-1)i}$$
$$+ a(A_i + A_{3i}) + b(A_i + A_{5i}) + c(A_{3i} + A_{7i}) + \ldots + g(A_{(2k-3)i} + A_{(2k+1)i})$$
$$= (3a + b)\,A_i + (a + 2b + c)\,A_{3i} + (b + 2c + d)\,A_{5i} + (c + 2d + e)\,A_{7i} + \ldots.$$

$$\therefore\ s_{2k-1} = a + b + c + d + \ldots + g\,, \qquad s_{2k+1} = 4(a + b + c + d + \ldots + g).$$

$$\therefore\ s_{2k+1} = 4s_{2k-1} = 2^4 s_{2k-3} = 2^6 s_{2k-5} = \ldots = 2^{2k} s_1 = 2^{2k}\,.$$

THEOREM. *For the sum of like even powers of the roots,* $s_{2k} = a_{2k-1}n - 2^{2k-1}$, *where* a_{2k-1} *is the coefficient of* A_i *in the expansion of* A_i^{2k-1}. (6)

$$A_i^{2k} = A_i(A_i^{2k-1})$$
$$= a(2 + A_{2i}) + b(A_{2i} + A_{4i}) + c(A_{4i} + A_{6i}) + \ldots + g(A_{(2k-2)i} + A_{2ki})$$
$$= 2a + (a + b)A_{2i} + (b + c)A_{4i} + (c + d)\,A_{6i} + \ldots + (f + g)A_{(2k-2)i} + gA_{2ki}.$$

$$\therefore\quad s_{2k} = 2ap - (a + b) - (b + c) - (c + d)\ldots - (f + g) - g$$
$$= 2ap + a - 2(a + b + c + d + \ldots + g)\,.$$

But
$$(a + b + c + d + \ldots + g) = s_{2k-1} = 2^{2k-2}\,;$$

$$\therefore\quad s_{2k} = 2ap + a - 2^{2k-1} = an - 2^{2k-1}\,.$$

Another proof of (6), which is needed below, is as follows:

Write $$A_i^{2k} = a + BA_{2i} + CA_{4i} + DA_{6i} + \ldots + RA_{2ki}.$$

Then $$s_{2k} = ap - (B + C + D + \ldots + R).$$

$$A_i^{2k+1} = aA_i + B(A_i + A_{3i}) + C(A_{3i} + A_{5i}) + \ldots$$
$$= (a + B)A_i + (B + C)A_{3i} + (C + D)A_{5i} + \ldots + RA_{(2k+1)i}.$$

$$\therefore\ s_{2k+1} = a + 2(B + C + D + \ldots + R) = a + 2ap - 2s_{2k} = na - 2s_{2k}.$$

But $a_{2k} = 2a_{2k-1}$, and $s_{2k+1} = 2^{2k}$; $\therefore\ s_{2k} = a_{2k-1}\,n - 2^{2k-1}$.

We may state this formula thus:

$$2s_{2k} = na_{2k} - 2^{2k}. \tag{7}$$

An interesting relation is derived as follows:

$$A_i^{2k+2} = (2 + A_{2i})A_i^{2k}$$
$$= 2a + 2BA_{2i} + 2CA_{4i} + \ldots + aA_{2i} + B(2 + A_{4i}) + C(A_{2i} + A_{6i}) + \ldots$$
$$= 2(a + B) + (a + 2B + C)A_{2i} + (B + 2C + D)A_{4i}$$
$$+ (C + 2D + E)A_{6i} + \ldots + (Q + 2R)A_{2ki} + RA_{(2k+2)i}.$$

$$\therefore\ s_{2k+2} = 2(a + B)p - (a + 2B + C) - (B + 2C + D) - (C + 2D + E) - \ldots$$
$$- (Q + 2R) - R$$
$$= 2(a + B)p - a + B - 4(B + C + D + \ldots + Q + R)$$
$$= 2(a + B)p - a + B + 4s_{2k} - 4ap$$
$$= -2ap + 2Bp - a + B + 4s_{2k} = n(B - a) + 4s_{2k}.$$

$$\therefore\ s_{2k+2} - 4s_{2k} = n(B_{2k} - a_{2k}). \tag{8}$$

To obtain the value of a_{2k-1}, break a Pascal Triangle along the diagonal indicated in Table 1 by the heavy figures; discard the part to the right, and turn the part to the left over. Thus Table 2, aside from this diagonal, gives the coefficients in the linear expansion of A_i^m. The reason why these coefficients obey the law of Pascal's Triangle $C_{p+1}^{q+1} = C_p^{q+1} + C_p^q$ is found in the proofs of (6) and (7).

C	0	1	2	3	4	5	6	7	8	9	... q
0	1										
1	1	**1**									
2	1	2	1								
3	1	3	**3**	1							
4	1	4	6	4	1						
5	1	5	10	**10**	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	**35**	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	**126**	84	36	9	1	
...											
p											

TABLE 1.

5	4	3	2	1	0	C
					1	0
				1	1	1
				2	1	2
			3	3	1	3
			6	4	1	4
		10	10	5	1	5
		20	15	6	1	6
	35	35	21	7	1	7
	70	56	28	8	1	8
126	126	84	36	9	1	9
252	210	120	45	10	1	10

TABLE 2.

Since the mth term of the series 1, 3, 10, 35, 126, ... (denoted by a_{2m-1}) is at the intersection of column m with row $(2m-1)$, it is

$$(2m-1)(2m-2)(2m-3)\ldots(2m-m)/(1.2.3\ldots m);$$

$$\therefore \quad a_{2m-1} = \frac{(2m-1)(2m-2)\ldots(m+1)}{1.2.3\ldots(m-1)} = \frac{(2m)!}{2.(m!)^2}. \tag{9}$$

Thus,

$$a_{2m-3} = \frac{(2m-3)(2m-4)\dots(m)}{1 \cdot 2 \dots (m-2)}; \quad \therefore\ ma_{2m-1} = 2(2m-1)\,a_{2m-3}.$$

It follows that

$$a_{2m-1} = \frac{(2m-1)(2m-3)(2m-5)\dots 3}{1 \cdot 2 \cdot 3 \dots m} \cdot 2^{m-1}.$$

We have thus proven that

$$s_{2k+1} = 2^{2k}; \quad s_{2k} = \frac{(2k-1)(2k-2)\dots(k+1)}{1 \cdot 2 \cdot 3 \dots (k-1)}\, n - 2^{2k-1}. \tag{10}$$

The coefficients of n are 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, etc.

The general coefficient C_m in the equation sought, expressed in terms of the sums of like powers of the roots, is

$$C_m = \Sigma \frac{(-1)^{t_1+t_2+\dots+t_m}\, s_1^{\,t_1} s_2^{\,t_2} \dots s_m^{\,t_m}}{\pi(t_1+1)\,\pi(t_2+1)\dots\pi(t_m+1)\,2^{t_2}3^{t_3}\dots m^{t_m}}, \tag{11}$$

where $t_1, t_2, \dots, t_m$ take all positive values, including zero, which satisfy the equation

$$t_1 + 2t_2 + 3t_3 + \dots + mt_m = m, \tag{12}$$

and $\pi(t_i + 1) = 1 \cdot 2 \cdot 3 \dots t_i$, with the assumption $\pi(1) = 1$.

In our problem,

$$s_1^{\,t_1} s_2^{\,t_2} \dots s_m^{\,t_m} = (n-2)^{t_2} \cdot 2^{2t_3} \cdot (3n-2^3)^{t_4} \cdot 2^{4t_5} \dots$$

$$= 2^{(2t_3+4t_5+6t_7+\dots+2kt_{2k+1}+\dots)}(n-2)^{t_2}(3n-2^3)^{t_4}(10n-2^5)^{t_6}\dots$$

$$\times \left[\frac{(2k-1)(2k-2)\dots(k+1)}{1 \cdot 2 \cdot 3 \dots (k-1)}\, n - 2^{2k-1}\right]^{2k} \dots .$$

Thus,

$$C_3 = \frac{-2^2}{3} + \frac{n-2}{2} - \frac{1}{2 \cdot 3} = \frac{n-5}{2} = (p-2)$$

$$C_4 = \frac{-(3n-8)}{4} + \frac{2^2}{3} - \frac{n-2}{4} + \frac{(n-2)^2}{2 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4}$$

$$= \tfrac{1}{8}(n-5)(n-7) = \tfrac{1}{2}(p-2)(p-3).$$

We may determine these coefficients by the equation

$$(-1)^m m!\,C_m = \begin{vmatrix} s_1 & 1 & 0 & 0 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & 0 & 0 & \dots & 0 \\ s_3 & s_2 & s_1 & 3 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overset{m-1}{\cdot} \\ s_m & s_{m-1} & s_{m-2} & s_{m-3} & \dots & \dots & s_1 \end{vmatrix} \tag{13}$$

$$= \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ n-2 & 1 & 2 & 0 & 0 & 0 & \dots \\ 2^2 & n-2 & 1 & 3 & 0 & 0 & \dots \\ 3n-2^3 & 2^2 & n-2 & 1 & 4 & 0 & \dots \\ 2^4 & 3n-2^3 & 2^2 & n-2 & 1 & 5 & \dots \\ 10n-2^5 & 2^4 & 3n-2^3 & 2^2 & n-2 & 1 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

Multiplying each row by 2 and adding to the one below,

$$(-1)^m m!\,C_m = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ n & 3 & 2 & 0 & 0 & 0 & \dots \\ 2n & n & 5 & 3 & 0 & 0 & \dots \\ 3n & 2n & n & 7 & 4 & 0 & \dots \\ 6n & 3n & 2n & n & 9 & 5 & \dots \\ 10n & 6n & 3n & 2n & n & 11 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \tag{13$_a$}$$

By either method we obtain the equation whose roots are the chords A_1, $-A_2$, A_3, $-A_4$, ..., $-(-1)^p A_p$ of the circle of unit radius, viz.,

$$x^p - x^{p-1} - (p-1)x^{p-2} + (p-2)x^{p-3} + \frac{(p-2)(p-3)}{1.2}x^{p-4}$$

$$-\frac{(p-3)(p-4)}{1.2}x^{p-5} - \dots$$

$$+(-1)^m \frac{(p-m)(p-m-1)\dots(p-2m+1)}{1.2.3\dots m}x^{p-2m}$$

$$-(-1)^m \frac{(p-m-1)(p-m-2)\dots(p-2m)}{1.2.3\dots m}x^{p-2m-1} - \dots = 0. \tag{14}$$

I will not enter into the decomposition of this equation. Its irreducibility in the domain of rational numbers follows from Eisenstein's Theorem.

For regular polygons of a composite number of sides $n = rm$, we have the fundamental theorem

$$A_s - A_{m-s} - A_{m+s} + A_{2m-s} + A_{2m+s} - \dots$$
$$+ (-1)^{\frac{1}{2}(r-1)} (A_{\frac{1}{2}(r-1)m-s} + A_{\frac{1}{2}(r-1)m+s}) = 0 . \qquad (15)$$

s being any integer $\overline{\overline{<}} \frac{1}{2}(m-1)$.

PROOF: $A_m - A_{2m} + A_{3m} - A_{4m} + \dots - (-1)^{\frac{1}{2}(r-1)} A_{\frac{1}{2}(r-1)m} = 1$, since A_m, etc. are chords of the regular r-gon. Hence

$$A_s = A_s (A_m - A_{2m} + A_{3m} - \dots)$$
$$= A_{m-s} + A_{m+s} - A_{2m-s} - A_{2m+s} + \dots .$$

To find the equation whose roots are A_s, $-A_{m-s}$, $-A_{m+s}$, A_{2m-s}, A_{2m+s}, $\dots (-1)^{\frac{1}{2}(r-1)} A_{\frac{1}{2}(r-1)m\pm s}$, we proceed as in the case of a regular polygon of a prime number of sides. Evidently the coefficients in the expansion of $A^l_{km\pm s}$ are the same as those found for A^l_i. But in taking the sum of the r expressions for the like powers of the r roots, we have the sum of the roots $= 0$ now. Further, since n is a composite number, we may find in the expansion of $A^l_{km\pm s}$ chords which belong also to a regular polygon the number of whose sides is a divisor of n. In that case, in summing we obtain sub-groups of the chords instead of the entire groups (15).

Thus, for a regular polygon of 35 sides, take $m = 7$, $r = 5$;

$$\therefore \quad A_1 - A_6 - A_8 + A_{13} + A_{15} = 0 .$$

$$S_2 = A_1^2 + A_6^2 + A_8^2 + A_{13}^2 + A_{15}^2$$
$$= 10 + (A_2 + A_{12} + A_{16} - A_9 - A_5) = 10 \text{ by (15).}$$

$$S_3 = A_1^3 - A_6^3 - A_8^3 + A_{13}^3 + A_{15}^3$$
$$= 3(A_1 - A_6 - A_8 + A_{13} + A_{15}) + (A_3 + A_{17} + A_{11} - A_4 - A_{10}) = 0 .$$

$$S_4 = \Sigma_i (6 + 4A_{2i} + A_{4i})$$
$$= 5.6 + 4(A_2 + A_{12} + A_{16} - A_9 - A_5) + (A_4 - A_{11} - A_3 - A_{17} + A_{10}) = 5.6.$$

But

$$S_5 = \Sigma_i (10A_i + 5A_{3i} + A_{5i})$$
$$= 10(A_1 - A_6 - A_8 + A_{13} + A_{15}) + 5(A_3 + A_{17} + A_{11} - A_4 - A_{10})$$
$$+ 5A_5 = 5A_5 .$$

Similarly,

$$S_7 = 35(A_1 - A_6 \ldots) + 21(A_3 + A_{17} \ldots) + 35A_5 + (2A_7 - 2A_{14} - 2)$$
$$= 35A_5,$$

since $A_7 - A_{14} = 1$, by (4), being chords of the pentagon.

For a regular 105-gon, take $m = 7$, $r = 15$. Then

$$A_1 - A_6 - A_8 + A_{13} + A_{15} - A_{20} - A_{22} + A_{27} + A_{29} - A_{34} - A_{36} + A_{41}$$
$$+ A_{43} - A_{48} - A_{50} = 0.$$

$$S_2 = 30 + A_2 - A_5 - A_9 + A_{12} + A_{16} - \ldots$$
$$= 30.$$

$$S_3 = 3(A_1 - A_6 - A_8 + A_{13} + \ldots) + 3(A_3 - A_{18} - A_{24} + A_{39} + A_{45}).$$

But applying (15) for $s = 3$, $m = 21$,

$$A_3 - A_{18} + A_{24} + A_{39} + A_{45} = 0. \quad \therefore S_3 = 0.$$

$$S_4 = 6.15 + 4(A_2 - A_5 - A_9 + A_{12} + A_{16} \ldots) - (A_3 - A_4 - A_{10} + A_{11} + \ldots)$$
$$= 6.15.$$

$$S_5 = 10(A_1 - A_6 - A_8 + \ldots) + 15(A_3 - A_{18} - A_{24} + A_{39} + A_{45})$$
$$+ 5(A_5 - A_{30} - A_{40}).$$

But $$A_s - A_{35-s} - A_{35+s} = 0; \quad \therefore S_5 = 0.$$

$$S_6 = 15.20 + 15(A_2 - A_5 - A_9 + \ldots) - 6(A_3 - A_4 - A_{10} + \ldots)$$
$$+ 3(A_6 + A_{36} + A_{48} - A_{27} - A_{15}) = 15.20.$$

If the groups occurring in the expansion of S are all of the form (15), and hence zero, it is necessary to examine in S_{k+2} only the last group, viz., that arising from $\Sigma_i A_{(k+2)i}$, where i takes the r values $s, m - s, m + s, 2m - s, \ldots$; for all the previous groups are the same as those in S_k.

Thus in S_7, $\Sigma A_{7i} = 2(A_7 - A_{14} + A_{21} - A_{28} + A_{35} - A_{42} + A_{49}) - 2 = 0$ by (4); in S_9, $\Sigma A_{9i} = 3(A_9 - A_{12} - A_{30} + A_{33} - A_{51}) = 0$ by (15), for $s = 9$, $m = 21$; in S_{11}, $\Sigma A_{11i} = A_{11} + A_{39} + A_{17} - A_{38} + A_{45} - A_{10} - A_{32} - A_{18} - A_4 - A_{46} - A_{24} + A_{31} - A_{52} + A_3 + A_{25} = 0$, by (15), for $s = 3$, $m = 7$; similarly $\Sigma A_{13i} = 0$. But in S_{15}, $\Sigma A_{15i} = 15A_{15}$.

$$\therefore S_7 = S_9 = S_{11} = S_{13} = 0; \; S_{15} = 15A_{15}.$$

In forming the sum of like powers of the chords in the group (15), the common exponent being *prime* to n, we get, besides the numerical term, multiples of groups of the same form (15).

For the sum of like powers of the chords, the common exponent being $< r$, but not prime to n, we get, besides multiples of groups (15), multiples of sub-groups of the form

$$A_s - A_{km-s} - A_{km+s} + A_{2km-s} + A_{2km+s} - \ldots + (-1)^{\frac{1}{2}\left(\frac{r}{k}-1\right)} A_{\frac{1}{2}\left(\frac{r}{k}-1\right)m \pm s} = 0 .$$

Hence
$$S_{2l} = 2a_{2l-1}r , \tag{16}$$
in which a_{2l-1} is given by (9). But when this exponent $= r$, the last term ΣA_{rt} in the expansion of ΣA_i^r is

$$\begin{aligned} &A_{rs} - A_{rm-rs} - A_{rm+rs} + A_{2rm-rs} + A_{2rm+rs} - \ldots \\ &= A_{rs} + A_{rs} + A_{rs} + A_{rs} + A_{rs} + \ldots \\ &- rA_{rs} . \end{aligned}$$

$\therefore\ S_{2l+1} = 0$, if $2l + 1 < r$; while
$$S_r = rA_{rs} . \tag{17}$$

It follows from (11) that, in the equation sought, every coefficient k_{2l+1} (if $2l + 1 < r$) equals 0. For by (12)

$$t_1 + 2t_2 + 3t_3 + 4t_4 + \ldots + (2l+1)\, t_{2l+1} = 2l + 1 .$$

Hence at least one of the integers $t_1, t_3, t_5, \ldots t_{2l+1}$ must be an *odd* number > 0. Hence at least one factor under the summation sign in (11) will be zero. To obtain k_r, r being odd, we note that every term of the summation (11) in which any one of the integers $t_1, t_3, t_5, \ldots, t_{r-2}$ is different from zero will vanish. Thus for the remaining terms $2t_2 + 4t_4 + 6t_6 + \ldots + rt_r = r$; whence t_r is not 0, and therefore

$$t_2 = t_4 = t_6 = \ldots = t_{r-1} = 0 , \ t_r = 1 .$$

$$\therefore\ k_r = -\frac{rA_{rs}}{r} = -A_{rs} .$$

Similarly, for k_{2l} we may write $t_1 = t_3 = t_5 = \ldots = t_{2l-1} = 0$. Whence we obtain by an easy reduction

$$k_{2l} = \Sigma \frac{(-r)^{t_2+t_4+t_6+\ldots+t_{2l}}\, 3^{t_4} . 10^{t_6} . 35^{t_8} \ldots a_{2l-1}^{t_{2l}}}{\pi(t_2+1)\, \pi(t_4+1) \ldots \pi(t_{2l}+1)\, 2^{t_4} . 3^{t_6} . 4^{t_8} \ldots l^{t_{2l}}} , \tag{18}$$

where
$$t_2 + 2t_4 + 3t_6 + \ldots + lt_{2l} = l .$$

Thus
$$k_2 = -r ;$$

$$k_4 = \frac{(-r)^2}{2} + \frac{3(-r)}{2} = \frac{r(r-3)}{1 . 2} ;$$

$$k_6 = \frac{(-r)^3}{2 . 3} + \frac{3(-r)^2}{2} + \frac{10(-r)}{3} = \frac{-r(r-4)(r-5)}{1 . 2 . 3} .$$

Substituting the values given by (16) and (17) in (13), we obtain

$$(2l)!\,k_{2l} = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 2r & 0 & 2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 2r & 0 & 3 & 0 & 0 & \dots & 0 \\ 6r & 0 & 2r & 0 & 4 & 0 & \dots & 0 \\ 0 & 6r & 0 & 2r & 0 & 5 & \dots & 0 \\ 20r & 0 & 6r & 0 & 2r & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \underset{2l-1}{\cdot} \\ 2a_{2l-1}r & 0 & 2a_{2l-3}r & 0 & 2a_{2l-5}r & 0 & \dots & 0 \end{vmatrix}$$

Striking out the 1st row and 2d column; then the (present) 3d row and 4th column; then the 5th row and 6th column, etc., we have

$$= (-1)^l . 1 . 3 . 5 . 7 \dots (2l-1) \begin{vmatrix} 2r & 2 & 0 & 0 & \dots & 0 \\ 6r & 2r & 4 & 0 & \dots & 0 \\ 20r & 6r & 2r & 6 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2a_{2l-1}r & 2a_{2l-3}r & \dots & & \dots & 2r \end{vmatrix};$$

$$\therefore\ (-1)^l\, l!\, k_{2l} = \begin{vmatrix} r & 1 & 0 & 0 & 0 & \dots & 0 \\ 3r & r & 2 & 0 & 0 & \dots & 0 \\ 10r & 3r & r & 3 & 0 & \dots & 0 \\ 35r & 10r & 3r & r & 4 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \underset{l-1}{\cdot} \\ a_{2l-1}r & a_{2l-3}r & a_{2l-5}r & \dots & \dots & \dots & r \end{vmatrix}. \qquad 19)$$

By either method we obtain the equation whose roots are the chords A_s, $-A_{m-s}$, $-A_{m+s}$, A_{2m-s}, A_{2m+s}, ..., $(-1)^{\frac{1}{2}(r-1)} A_{\frac{1}{2}(r-1)m\pm s}$; viz.,

$$x^r - rx^{r-2} + \frac{r(r-3)}{1.2} x^{r-4} - \frac{r(r-4)(r-5)}{1.2.3} x^{r-6} + \frac{r(r-5)(r-6)(r-7)}{1.2.3.4} x^{r-8}$$

$$- \frac{r(r-6)(r-7)(r-8)(r-9)}{1.2.3.4.5} x^{r-10} + \dots + (-1)^{\frac{1}{2}(r-1)} . rx - A_{rs} = 0. \qquad (20)$$

If r be prime to m, the regular polygon of mr sides depends for inscription upon the same equations as does the regular r-gon, together with equations whose degrees are given by the prime factors of $m-1$. If, however, r contains the factor m, the regular polygon of mr sides depends for inscription upon the same equations as does the regular r-gon, together with an equation of the mth degree of the form (20).

Compare Art. 109 of Serret's *Cours d'Algèbre Supérieure.*

UNIVERSITY OF CHICAGO, *December*, 1894.